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Abstract

The special functions originated during the eighteenth and nineteenth centuries, from some problems of applied physics and engineering that lead to differential equations that do not have solutions in terms of well-known functions. The solutions to these problems were found in terms of some new functions that possessed interesting properties such as orthogonality, integral transformation, differentiation, convergence, etc. These functions have been given the name as special functions. Later on it was observed that most of the special functions can be represented in terms of hypergeometric functions and their generalizations.

Special functions have been used for several centuries, since they have numerous applications in astronomy, trigonometric functions which have been studied for over a thousand years. Even the series expansions for sine and cosine, as well as the arctangent were known for long time ago from the fourteenth century. Since then the subject of special functions has been continuously developed with contribution of several mathematicians including Euler, Legendre, Laplace, Gauss, Kummer, Riemann and Ramanujan. In the past several years, the discoveries of new special functions and applications of this kind of functions to new areas of mathematics have initiated a great interest of this field. The study of special functions grew up with the calculus and is consequently one of the oldest branches of analysis. It flourished in the 19th century as part of the theory of complex variables. In the second half of the 20th century it has received a new impetus from a connection with Lie groups and a connection with averages of elementary functions. The history of special functions is closely tied to the problem of terrestrial and celestial mechanics that were solved in the 18th and 19th centuries, the boundary value problems of electromagnetism and heat in the 19th and the eigenvalue problems of quantum mechanics in twentieth.

Seventeenth century, England was the birthplace of special functions. John Wallis at Oxford took two first steps towards the theory of Gamma function long before Euler reached it. Euler found most of the major properties of the Gamma functions around 1730. In 1772, Euler evaluated the Beta function integral in terms of the Gamma function. Only the duplication and multiplication theorems remained to be discovered by Legendre and Gauss respectively early in the next century. A

major development was the theory of hypergeometric series which began in a systematic way (although some important results had been found by Euler and Pfaff) with Gauss's memoir on the ${}_2F_1$ series in 1812, a memoir which was a landmark also on the path towards rigour in mathematics. The ${}_3F_2$ series was studied by Clausen (1832) and the ${}_1F_1$ series by Kümmer (1837). The functions which Bessel considered in his memoir of 1824 are ${}_0F_1$ series; Bessel started from a problem in orbital mechanics, but the functions have found a place in every branch of mathematical physics. Near the end of the century, Appell (1880) introduced hypergeometric functions of two variables and Lauricella generalized them to several variables in 1893.

In the present work, an attempt has been made to study the **DEMONSTRATION OF SUITABILITY OF HYPERGEOMETRIC TECHNIQUE IN ADDRESSING SIGNIFICANT PROBLEMS**. This thesis is spread over in **ten chapters**.

Chapter 1 is devoted to the brief survey of the literature of the subject. It aims at introducing several classes of special functions, which occur rather more frequently in the study of summations and transformations needed for the presentation of subsequent chapters.

In **Chapter 2**, we derive the new hypergeometric forms of some functions, like: $\left(\sqrt{(z-1)}-\sqrt{z}\right)^{b} \pm \left(\sqrt{(z-1)}-\sqrt{z}\right)^{-b}$ and $\left(\sqrt{(z-1)}+\sqrt{z}\right)^{b} \pm \left(\sqrt{(z-1)}+\sqrt{z}\right)^{-b}$, where b is an arbitrary parameter, in terms of Gauss' functions. As applications of these results, we obtain the hypergeometric forms of r, where r is the root of the cubic equation $r^3 - r + \frac{2}{3}\sqrt{\left(\frac{z}{3}\right)} = 0$, out of which two hypergeometric forms are new companions of an entry(68) on page(472) of Vol.III of the compendium of Prudnikov *et al.*.

In **Chapter 3**, we obtain the summations of some infinite series by using certain hypergeometric summation theorems of positive and negative unit arguments and other associated functions. We also obtain some hypergeometric summation theorems for:

$${}_8F_7\left[\tfrac{9}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},3,3,1;\tfrac{7}{2},\tfrac{7}{2},\tfrac{7}{2},\tfrac{7}{2},\tfrac{1}{2},2,2;1\right],$$

$${}_5F_4\left[\tfrac{5}{3},\tfrac{4}{3},\tfrac{4}{3},\tfrac{1}{3},\tfrac{1}{3};\tfrac{2}{3},1,2,2;1\right], \qquad {}_4F_3\left[\tfrac{3}{2},\tfrac{3}{2},1,1;\tfrac{5}{2},\tfrac{5}{2},2;1\right],$$

$${}_5F_4\left[\tfrac{9}{4},\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2};\tfrac{5}{4},2,3,3;1\right], \qquad {}_4F_3\left[\tfrac{2}{3},\tfrac{1}{3},1,1;\tfrac{7}{3},\tfrac{5}{3},2;1\right],$$

$${}_5F_4\left[\tfrac{13}{8},\tfrac{5}{4},\tfrac{5}{4},\tfrac{1}{4},\tfrac{1}{4};\tfrac{5}{8},2,2,1;1\right], \qquad {}_4F_3\left[\tfrac{7}{6},\tfrac{5}{6},1,1;\tfrac{13}{6},\tfrac{11}{6},2;1\right],$$

$${}_5F_4\left[\tfrac{1}{2},\tfrac{1}{2},\tfrac{5}{2},\tfrac{5}{2},1;\tfrac{3}{2},\tfrac{3}{2},\tfrac{7}{2},\tfrac{7}{2};-1\right], \qquad {}_4F_3\left[1,1,1,1;3,3,3;-1\right].$$

In **Chapter 4**, we obtain the semi-integrals of certain algebraic expressions in terms of complete elliptic integrals of different kinds by using Euler's linear transformation, Pfaff-Kümmer linear transformation and series manipulation technique.

In **Chapter 5**, we aim to consider certain generalized hypergeometric function ${}_3F_2$ with particular arguments, through which a number of summation formulas for ${}_{p+1}F_p(1)$ are provided. We then establish a power series whose coefficients are involved in generalized hypergeometric functions with unit argument. We also demonstrate that the generalized hypergeometric functions with unit argument mentioned before may be expressed in terms of Bell polynomials. Further we explore several special instances of our primary identities, among numerous others.

In **Chapter 6**, we introduce two sequences of *new* numbers and their derivatives, which are closely related to the Stirling numbers of the first kind, and choose to employ six known generalized Kummer's summation formulas for ${}_2F_1(-1)$ and ${}_2F_1(1/2)$, we establish six classes of generalized summation formulas for ${}_{p+2}F_{p+1}$ with arguments -1 and $1/2$ for any positive integer p. Next, by differentiating both sides of six chosen formulas presented with respect to a specific parameter, among numerous ones, we demonstrate six identities in connection with finite sums of ${}_4F_3(-1)$ and ${}_4F_3(1/2)$. Further, we choose to give simple particular identities of some formulas presented. We conclude the chapter by highlighting potential use of the newly presented numbers.

In **Chapter 7**, we obtain the solutions of Nielsen-type integrals and the other associated integrals through hypergeometric function approach. Some applications of Nielsen-type integrals are also demonstrated in the form of Weber-type and Anger-type functions.

In **Chapter 8**, the hypergeometric generating function with suitable convergence conditions in the form of Srivastava-Daoust triple hypergeometric function is derived by using series rearrangement technique. Some generating functions for Bedient's two polynomials and Shively pseudo-Laguerre polynomials are also obtained as special cases.

In the last two chapters; **Chapter 9** and **Chapter 10**, we derive the analytical expressions (not previously found and recorded in the literature) for the exact curved surface area of the hyperboloid of one sheet and two sheets. The derivation is based on Mellin-Barnes type contour integral representations of generalized hypergeometric function ${}_pF_q(z)$, Meijer's G-function, decomposition formula for Meijer's G-function and series rearrangement technique. Further, we also obtain the formula for the volume of the hyperboloid of one sheet and two sheets. The closed forms for the exact curved surface area and volume of the hyperboloid of one sheet and two sheets are also verified numerically by using *Mathematica Program.*